ONLINE APPENDIX:
On the band spectral estimation of business cycle models

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Appendix

A Appendix to Section 2

A.1 Angeletos et al. (2018) model

Table 1: Parameter values, ACD (2018) model

	parameter	posterior median
ψ	utilization elasticity	0.500
ν	inverse labor supply elasticity	0.282
α	capital share	0.255
φ	investment adjustment costs	3.312
b	habit persistence	0.758
χ	Calvo parameter	0.732
κ_R	Taylor rule smoothing	0.198
κ_π	Taylor rule inflation	2.271
κ_y	Taylor rule output	0.121
ρ_m	AR mon. policy	0.647
ρ_a	AR transitory TFP component	0.412
ρ_n	AR news	0.224
ρ_i	AR transitory investment-specific technology	0.374
ρ_c	AR preference	0.888
ρ_g	AR government spending	0.786
ρ_ξ	AR confidence	0.833
σ_a^P	std. permanent TFP component	0.406
σ_a^T	std. transitory TFP component	0.347
σ_n	std. news	0.378
σ_i^P	std. permanent investment-specific technology	0.610
σ_i^T	std. transitory investment-specific shocks	5.805
σ_c	std. preference	0.357
σ_g	std. government spending	1.705
σ_ξ	std. confidence	0.613
σ_m	std. mon. policy	0.313

B Appendix to Section 4

B.1 Q-Q plots

The Q-Q plots compare the quantiles of the sampling distributions of the estimated parameters to those of the standard normal distribution. Parameter estimates are normalized by subtracting their respective means and dividing by their respective standard deviations. The black lines represent the 45-degree line.

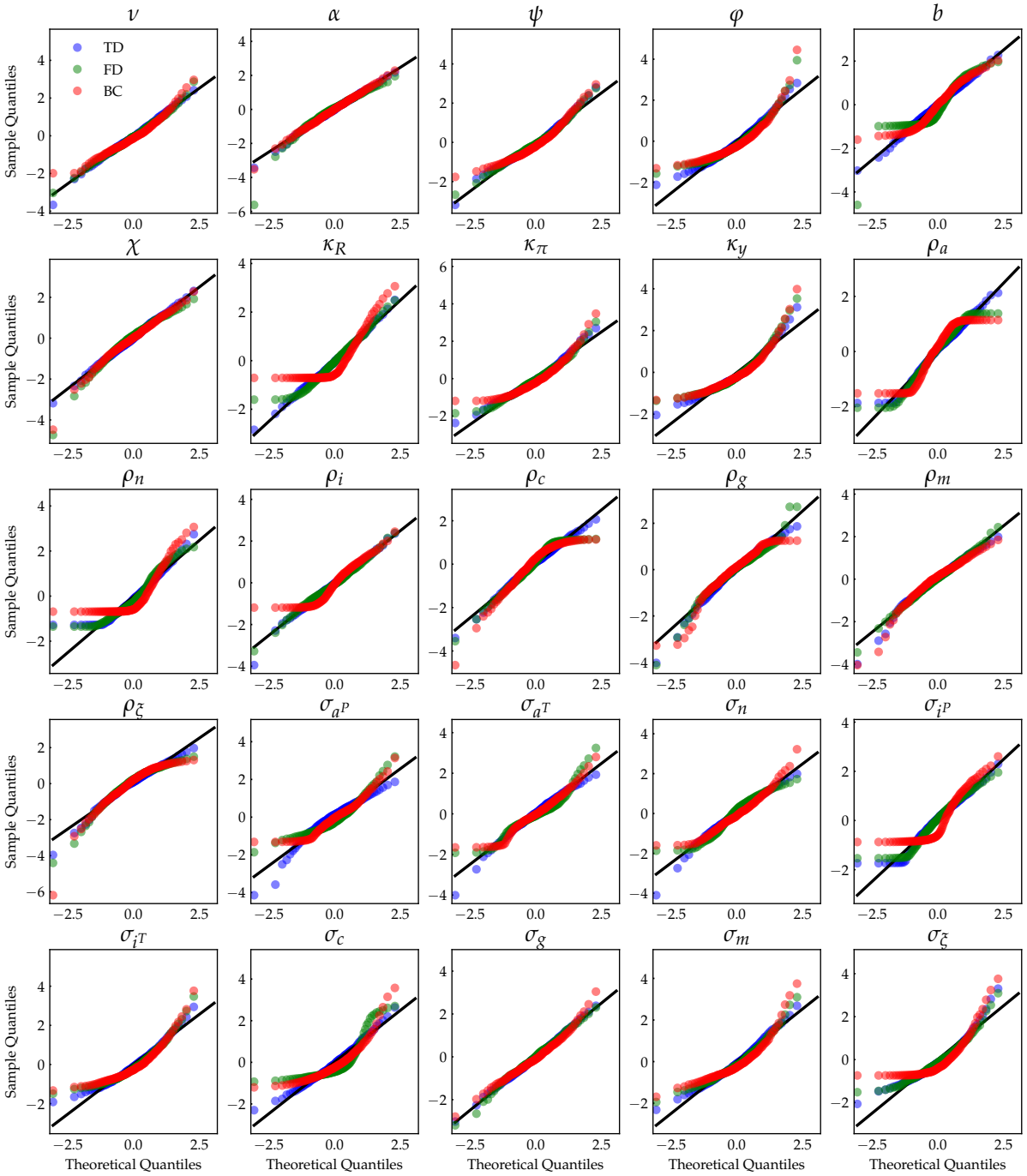


Figure 1: Baseline parametrization. The results are based on 1000 MC replications with a sample size of $T = 192$.

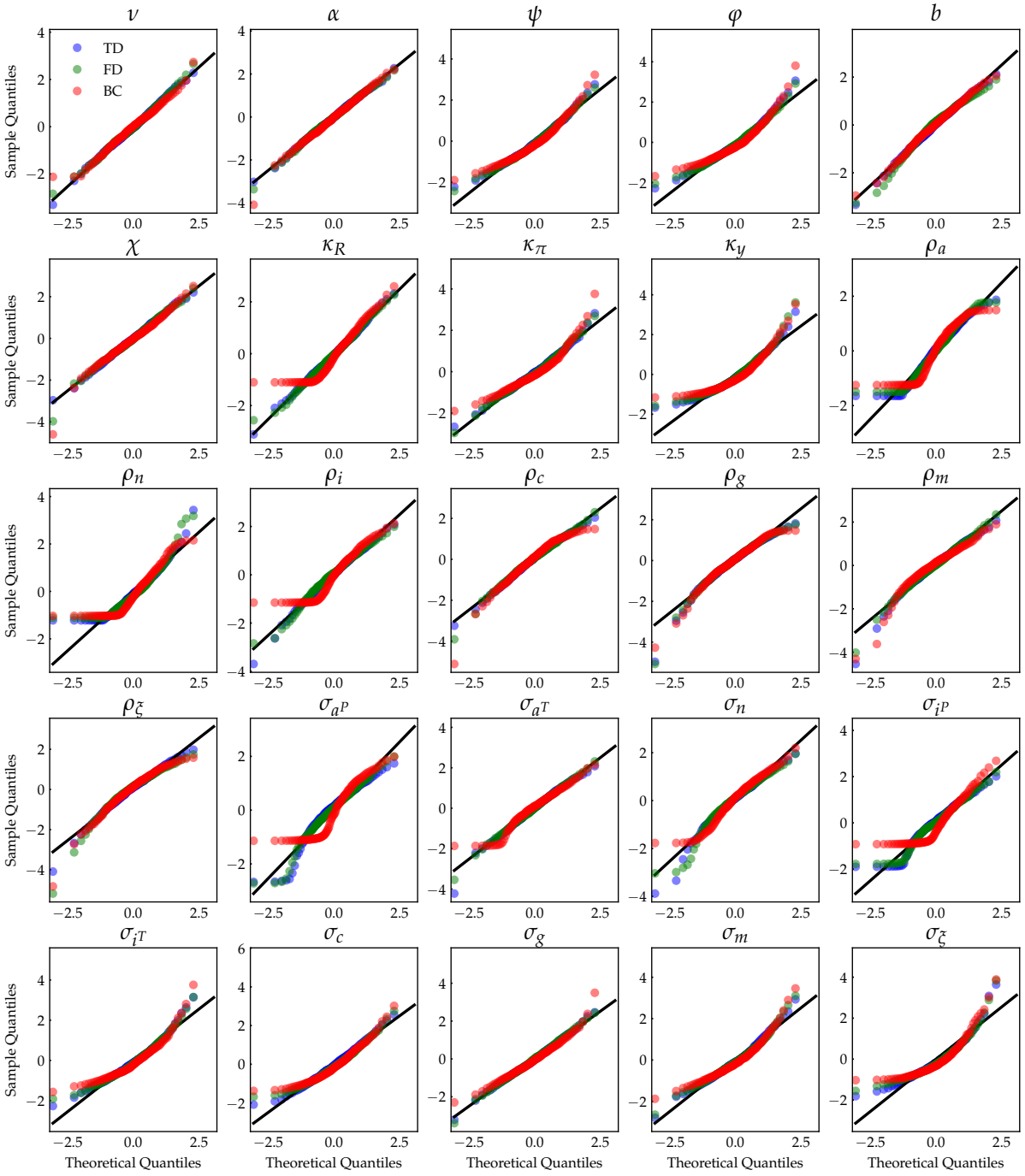


Figure 2: Alternative parametrization of the ACD model. The results are based on 1000 MC replications with a sample size of $T = 192$.

C Appendix to Section 5

This appendix contains additional results complementing Section 5 in the main text.

C.1 Results for the baseline parametrization

Table 2: Estimation uncertainty and efficiency loss: MC vs CRLBs

	estimation uncertainty					efficiency loss			
	all freqs			BC freqs		BC vs all freqs			
parameter	MC-TD	MC-FD	CRLB	MC	CRLB	MC-TD	MC-FD	CRLB	CRLB(∞)
ν	26.7	39.3	25.7	76.9	55.6	2.9	2.0	2.2	2.3
α	4.0	6.0	3.8	7.4	6.3	1.8	1.2	1.6	1.7
ψ	24.0	31.7	21.7	49.2	57.6	2.1	1.6	2.7	3.0
φ	26.2	44.7	24.8	48.8	42.6	1.9	1.1	1.7	1.8
b	4.2	13.7	4.0	13.1	7.6	3.1	1.0	1.9	2.0
χ	2.7	5.7	2.6	8.2	6.4	3.0	1.5	2.4	2.6
κ_R	35.0	44.3	33.8	76.2	118.6	2.2	1.7	3.5	3.7
κ_π	15.6	25.2	14.4	42.2	32.7	2.7	1.7	2.3	2.5
κ_y	48.2	131.8	44.1	96.7	106.2	2.0	0.7	2.4	3.0
ρ_a	54.4	71.0	57.1	91.4	185.6	1.7	1.3	3.3	3.4
ρ_n	77.4	126.2	85.7	116.3	279.1	1.5	0.9	3.3	3.4
ρ_i	19.8	30.0	19.7	65.7	63.7	3.3	2.2	3.2	3.4
ρ_c	3.5	5.7	3.5	7.1	8.9	2.0	1.2	2.5	3.2
ρ_g	7.3	11.4	6.2	28.1	15.4	3.9	2.5	2.5	2.8
ρ_m	8.8	11.1	8.0	23.9	18.1	2.7	2.1	2.3	2.4
ρ_ξ	4.8	12.4	4.4	13.1	11.4	2.7	1.1	2.6	3.0
σ_{a^P}	22.5	105.7	18.5	94.5	73.1	4.2	0.9	4.0	4.1
σ_{a^T}	19.2	55.3	18.4	71.0	66.2	3.7	1.3	3.6	3.7
σ_n	23.2	38.2	23.5	100.9	109.2	4.3	2.6	4.6	4.8
σ_{i^P}	56.7	78.1	44.1	122.5	208.5	2.2	1.6	4.7	6.1
σ_{i^T}	28.2	46.9	26.6	69.2	66.6	2.5	1.5	2.5	2.6
σ_c	42.2	301.9	44.1	136.6	86.0	3.2	0.5	2.0	2.2
σ_g	5.5	7.3	5.5	16.0	11.9	2.9	2.2	2.2	2.2
σ_m	11.3	17.6	11.2	34.3	25.0	3.0	1.9	2.2	2.4
σ_ξ	49.2	53.1	42.3	82.8	107.8	1.7	1.6	2.6	2.8

Note: Baseline parametrization. Estimation uncertainty is reported as the MC standard deviation or CRLB, expressed as percentage of the absolute value of the true parameter value. Efficiency loss is defined as the ratio of the standard deviations or CRLBs obtained using BC frequencies only vs. using all frequencies. CRLBs are computed with the exact FIM for $T = 192$ whereas CRLB(∞) is computed with the asymptotic FIM.

C.2 MC vs CRLBs: alternative parametrization

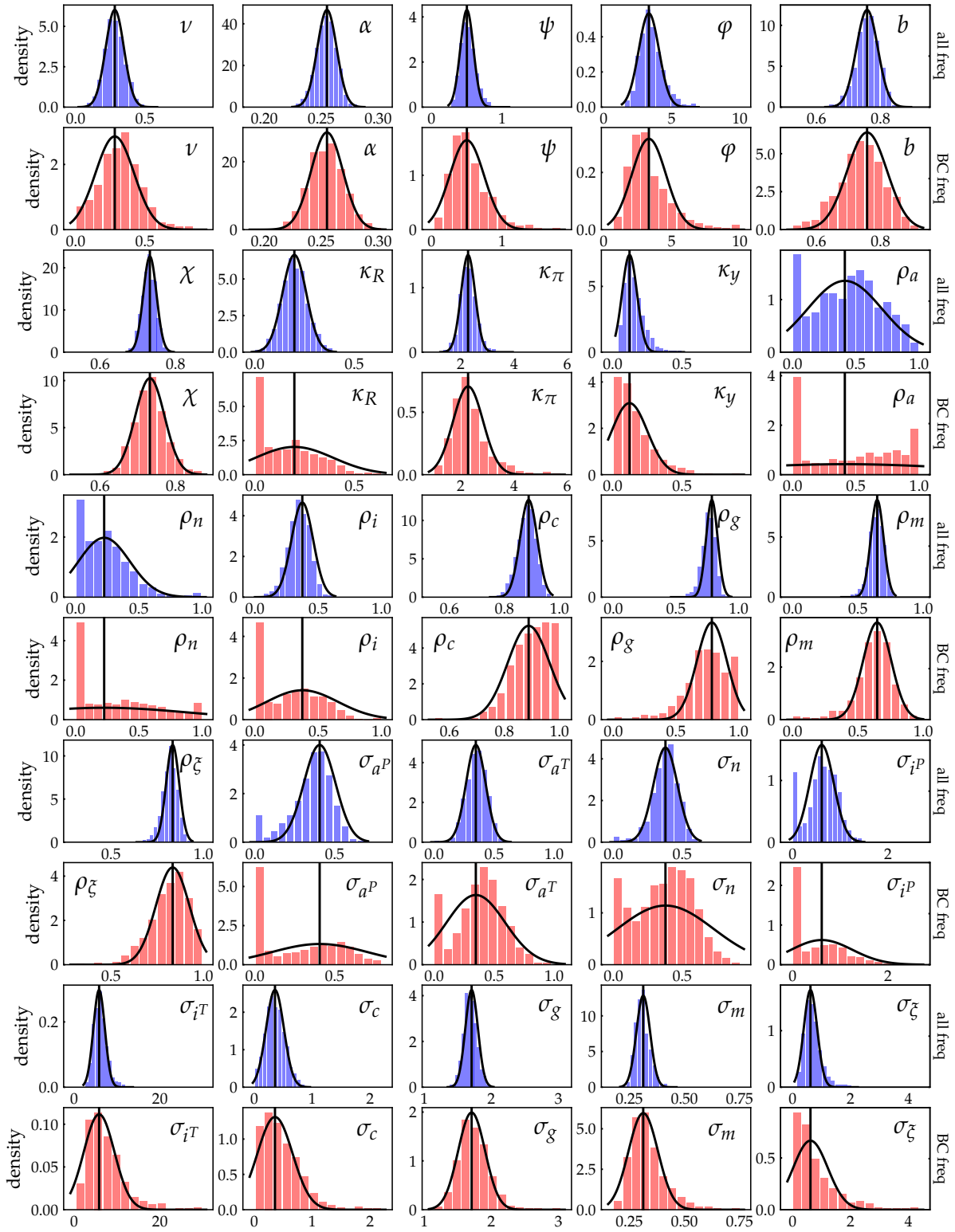


Figure 3: Each plot of the figure shows a histogram of the MC sampling distribution for the respective parameter, along with a gaussian density curve centered at the true parameter value and with a standard deviation equal to the CRLB. The sample size is $T = 192$.

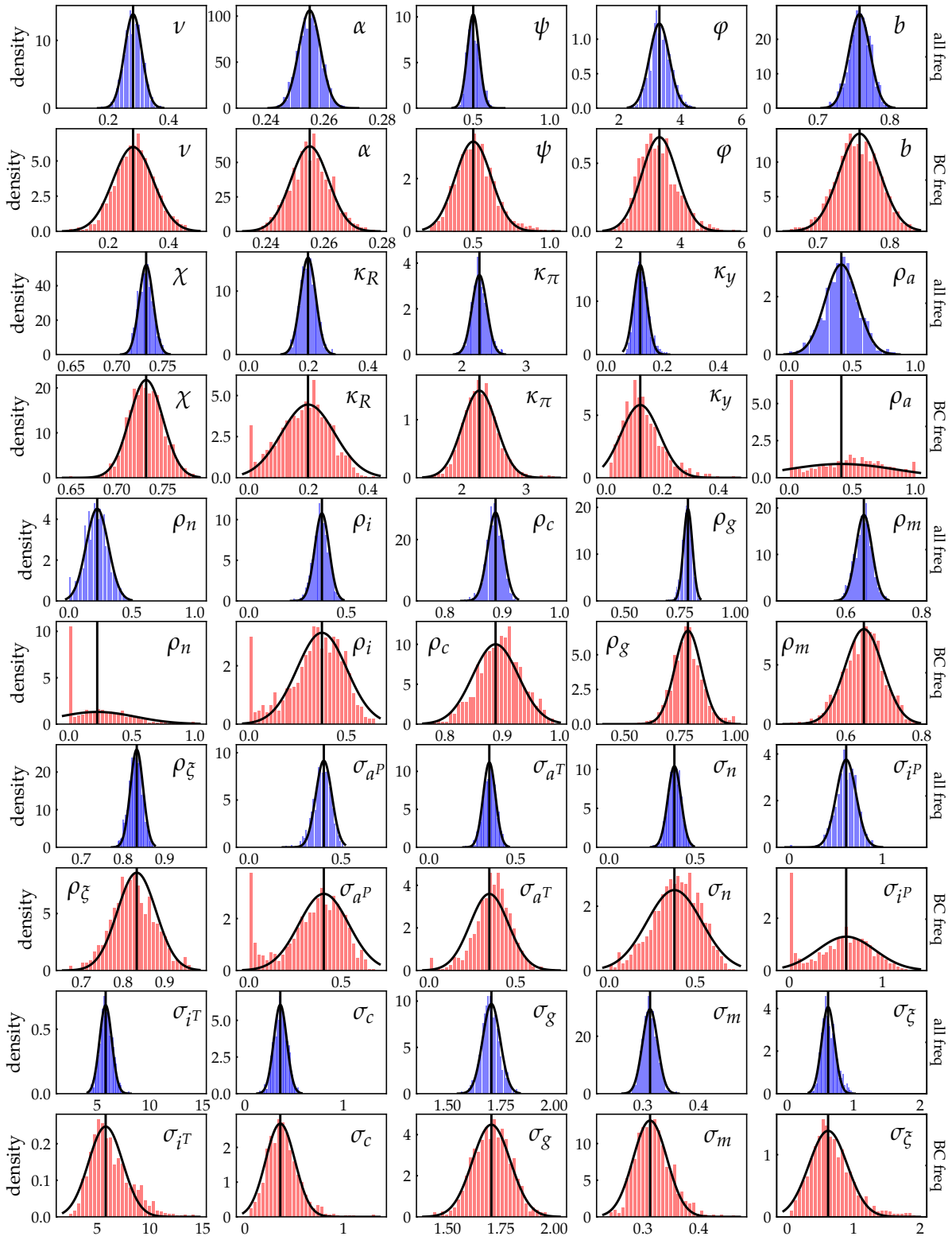


Figure 4: See the Notes to Figure 3. The sample size is $T = 1000$.

D Appendix to Section 6

This appendix contains additional results complementing Section 6 in the main text.

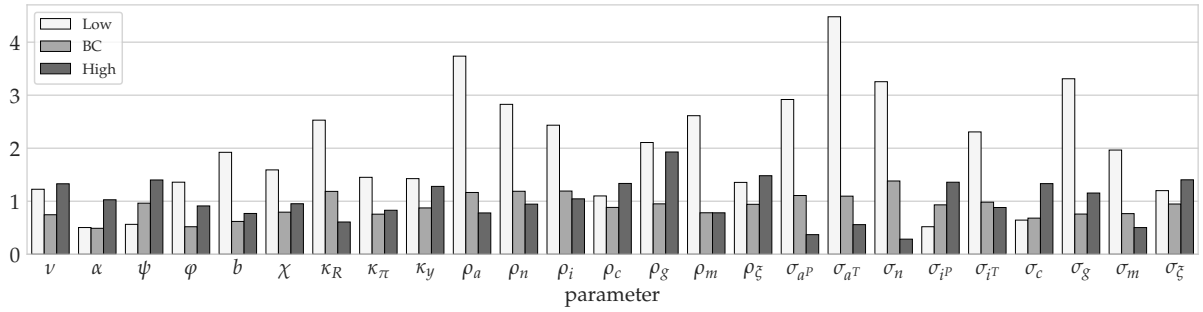


Figure 5: Log of the predicted relative efficiency of different band-spectral estimators. The relative efficiency is defined as the ratio of the CRLB value using a specific band of frequencies to the CRLB value using all frequencies. The sample size is $T = 192$.

Table 3: Decomposition of the predicted relative efficiency

parameter	Low	BC	High	BC+Low	BC+High
ν	$3.4 = 3.0 \times 1.13$	$2.1 = 1.7 \times 1.22$	$3.8 = 1.4 \times 2.80$	$1.3 = 1.5 \times 0.89$	$1.5 = 1.1 \times 1.37$
α	$1.7 = 2.3 \times 0.73$	$1.6 = 1.7 \times 0.94$	$2.8 = 1.4 \times 1.93$	$1.1 = 1.4 \times 0.81$	$1.4 = 1.1 \times 1.21$
ψ	$1.8 = 3.3 \times 0.52$	$2.6 = 1.8 \times 1.45$	$4.1 = 1.3 \times 3.15$	$1.2 = 1.6 \times 0.74$	$1.9 = 1.0 \times 1.77$
φ	$3.9 = 5.6 \times 0.69$	$1.7 = 1.8 \times 0.91$	$2.5 = 1.2 \times 2.03$	$1.3 = 1.7 \times 0.73$	$1.2 = 1.0 \times 1.19$
b	$6.8 = 4.9 \times 1.40$	$1.9 = 1.9 \times 0.98$	$2.2 = 1.2 \times 1.78$	$1.5 = 1.8 \times 0.85$	$1.2 = 1.0 \times 1.15$
χ	$4.9 = 3.6 \times 1.38$	$2.2 = 2.0 \times 1.12$	$2.6 = 1.2 \times 2.11$	$1.5 = 1.7 \times 0.88$	$1.3 = 1.0 \times 1.22$
κ_R	$12.5 = 5.5 \times 2.28$	$3.3 = 2.4 \times 1.34$	$1.8 = 1.1 \times 1.64$	$2.8 = 2.2 \times 1.24$	$1.1 = 1.0 \times 1.08$
κ_π	$4.3 = 4.2 \times 1.03$	$2.1 = 1.9 \times 1.11$	$2.3 = 1.2 \times 1.87$	$1.5 = 1.7 \times 0.86$	$1.3 = 1.0 \times 1.24$
κ_y	$4.2 = 4.4 \times 0.95$	$2.4 = 1.9 \times 1.27$	$3.6 = 1.2 \times 2.93$	$1.2 = 1.7 \times 0.71$	$1.9 = 1.0 \times 1.82$
ρ_a	$42.0 = 24.7 \times 1.70$	$3.2 = 3.2 \times 1.01$	$2.2 = 1.1 \times 2.06$	$2.7 = 3.1 \times 0.85$	$1.1 = 1.0 \times 1.13$
ρ_n	$16.9 = 2.2 \times 7.71$	$3.3 = 1.5 \times 2.15$	$2.6 = 1.6 \times 1.56$	$2.8 = 1.3 \times 2.20$	$1.2 = 1.1 \times 1.03$
ρ_i	$11.4 = 3.4 \times 3.35$	$3.3 = 1.2 \times 2.65$	$2.8 = 2.0 \times 1.44$	$2.8 = 1.2 \times 2.37$	$1.1 = 1.0 \times 1.05$
ρ_c	$3.0 = 3.1 \times 0.98$	$2.4 = 1.9 \times 1.29$	$3.8 = 1.3 \times 2.97$	$1.1 = 1.6 \times 0.70$	$2.0 = 1.1 \times 1.86$
ρ_g	$8.2 = 1.2 \times 6.74$	$2.6 = 2.3 \times 1.14$	$6.9 = 2.7 \times 2.55$	$1.3 = 1.1 \times 1.21$	$1.8 = 1.7 \times 1.05$
ρ_m	$13.6 = 12.0 \times 1.14$	$2.2 = 2.3 \times 0.95$	$2.2 = 1.1 \times 1.95$	$1.7 = 2.3 \times 0.77$	$1.2 = 1.0 \times 1.17$
ρ_ξ	$3.9 = 2.4 \times 1.59$	$2.6 = 1.9 \times 1.37$	$4.4 = 1.4 \times 3.25$	$1.3 = 1.5 \times 0.85$	$1.9 = 1.1 \times 1.74$
σ_{a^P}	$18.5 = 4.0 \times 4.59$	$3.0 = 1.9 \times 1.58$	$1.4 = 1.2 \times 1.18$	$2.8 = 1.7 \times 1.63$	$1.0 = 1.0 \times 1.01$
σ_{a^T}	$88.2 = 52.2 \times 1.69$	$3.0 = 3.1 \times 0.95$	$1.7 = 1.1 \times 1.65$	$2.8 = 3.1 \times 0.88$	$1.1 = 1.0 \times 1.07$
σ_n	$25.9 = 2.9 \times 9.04$	$4.0 = 1.5 \times 2.58$	$1.3 = 1.5 \times 0.90$	$3.5 = 1.4 \times 2.58$	$1.1 = 1.1 \times 0.99$
σ_{i^P}	$1.7 = 1.3 \times 1.34$	$2.5 = 2.1 \times 1.22$	$3.9 = 3.0 \times 1.29$	$1.1 = 1.1 \times 1.05$	$1.9 = 1.7 \times 1.11$
σ_{i^T}	$10.0 = 7.1 \times 1.41$	$2.7 = 1.9 \times 1.38$	$2.4 = 1.2 \times 2.03$	$2.3 = 1.9 \times 1.23$	$1.2 = 1.0 \times 1.16$
σ_c	$1.9 = 4.0 \times 0.47$	$2.0 = 1.8 \times 1.07$	$3.8 = 1.2 \times 3.04$	$1.1 = 1.7 \times 0.67$	$1.5 = 1.0 \times 1.47$
σ_g	$27.4 = 4.1 \times 6.74$	$2.1 = 1.9 \times 1.15$	$3.2 = 1.2 \times 2.56$	$2.1 = 1.7 \times 1.22$	$1.1 = 1.0 \times 1.06$
σ_m	$7.1 = 4.5 \times 1.58$	$2.2 = 1.9 \times 1.14$	$1.7 = 1.2 \times 1.35$	$1.8 = 1.7 \times 1.02$	$1.1 = 1.0 \times 1.11$
σ_ξ	$3.3 = 4.0 \times 0.83$	$2.6 = 1.8 \times 1.40$	$4.1 = 1.2 \times 3.26$	$1.3 = 1.7 \times 0.79$	$1.7 = 1.0 \times 1.68$

Note: The predicted relative efficiency is decomposed using (6.1) into a relative parameter sensitivity factor and a relative parameter interdependence factor.

D.1 Predicted parameter interdependence

Figures 6 – 10 display the highest pairwise and multiple correlation coefficients for the score of each parameter with the scores of 2 and 3 other parameters, as well as the multiple correlation coefficient with the scores of all other parameters. Results are presented for the low, BC, and high frequencies, as well as for the full spectrum.

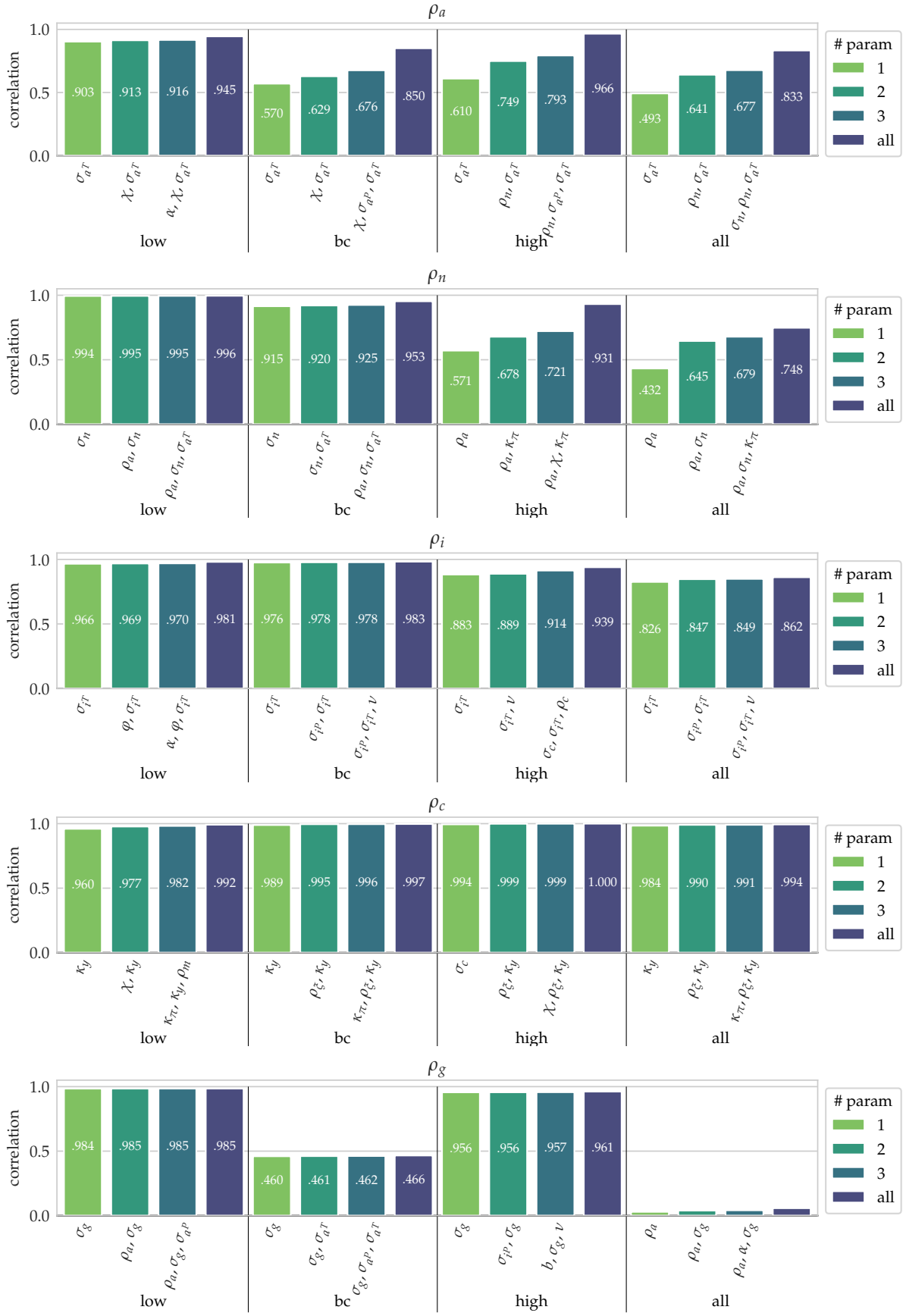


Figure 6: Highest score collinearity ($\max_j \{\mathrm{corr}\,(\frac{\partial \ell}{\partial \theta_i}, \frac{\partial \ell}{\partial \boldsymbol{\theta}_j})\}$) for different numbers of parameters ($\dim(\boldsymbol{\theta}_j)$ = 1, 2, 3, and 24 (labeled "all")) and different frequency bands ("low", "bc", "high", and "all").

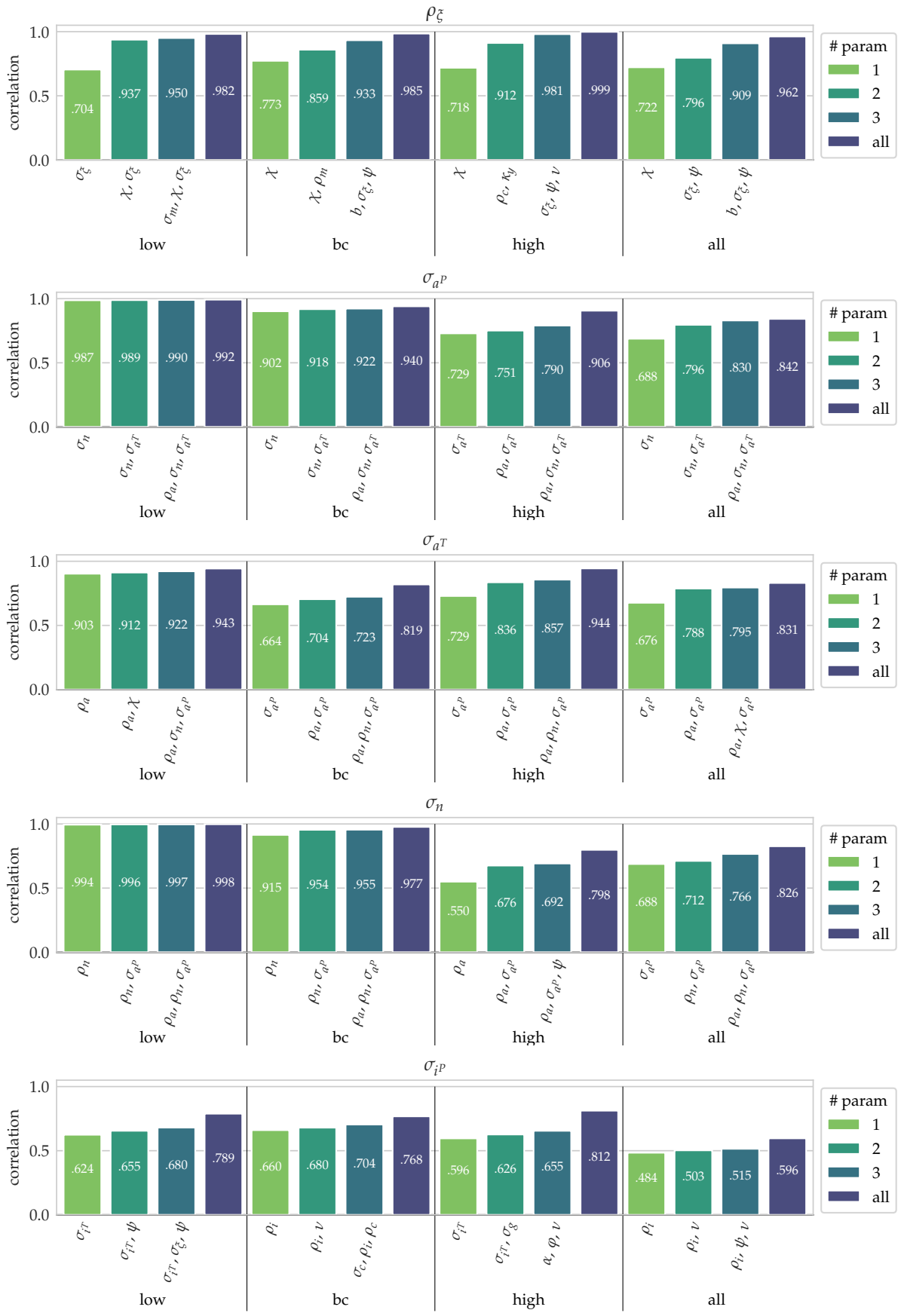


Figure 7: See Note to Figure 6.

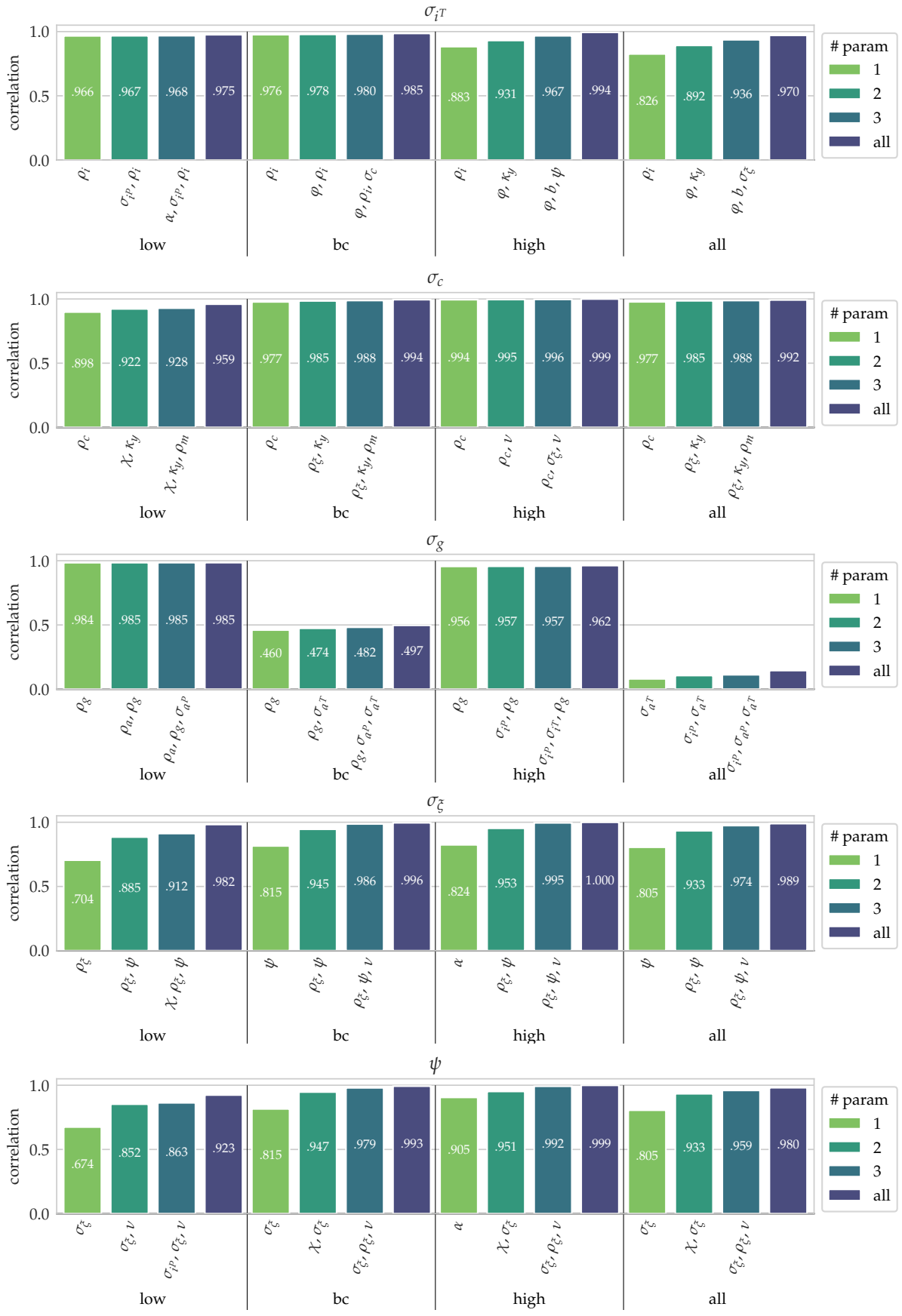


Figure 8: See Note to Figure 6.

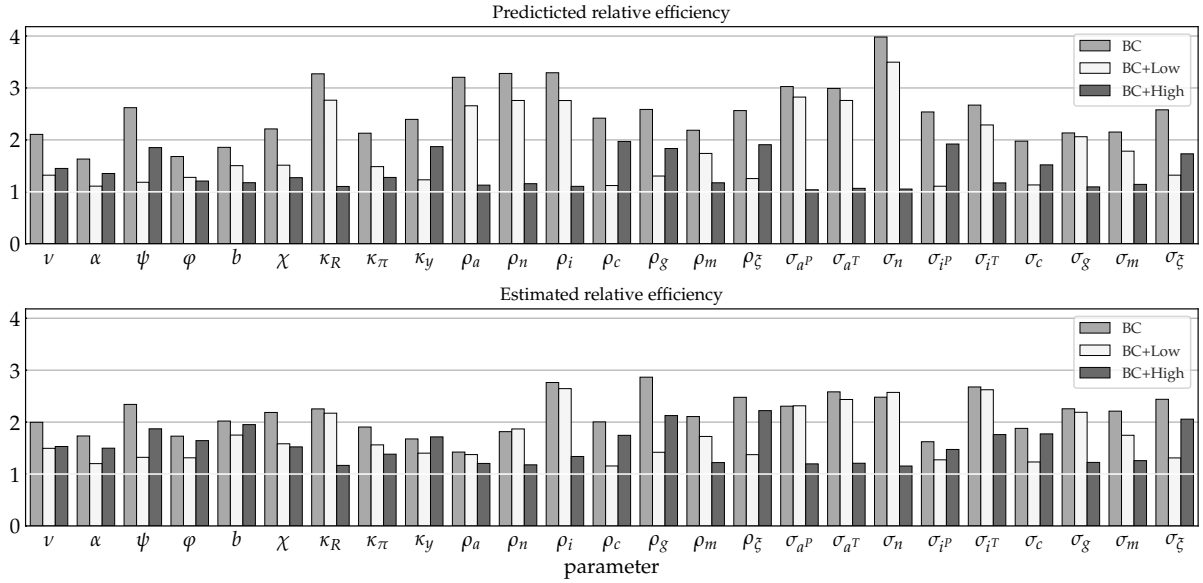
Figure 9: See Note to Figure 6.

Figure 10: See Note to Figure 6.

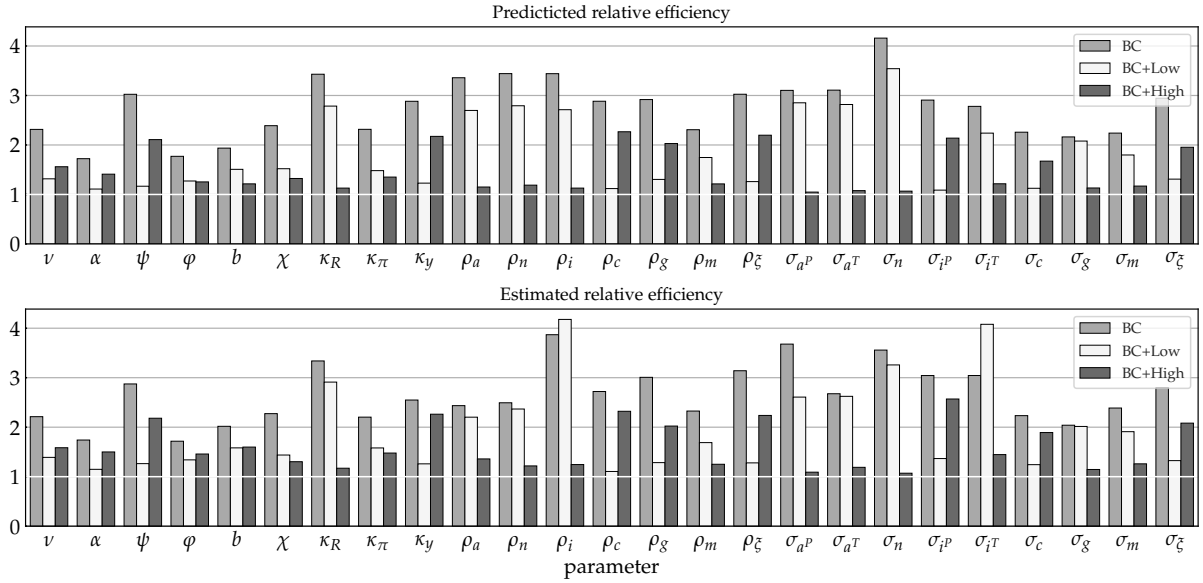
D.2 More Monte Carlo simulations

This section compares the CRLB-based predictions with the MC estimates of the efficiency loss for three band spectral estimators: using BC, BC+Low, and BC+High frequencies. Subsection D.2.1 presents the results for the case of joint estimation of all model parameters, while Subsection D.2.2 shows the results for the conditional case, where one parameter is estimated at a time.

D.2.1 Joint estimation: BC vs BC+Low vs BC+High



(a) T=192.



(b) T=1000.

Figure 11: Predicted (top panel of each subplot) and MC-estimated (bottom panel) relative efficiency of the three band spectral estimators. The relative efficiency is defined as the ratio of the MC standard deviation or CRLB using a band of frequencies to the value using all frequencies.

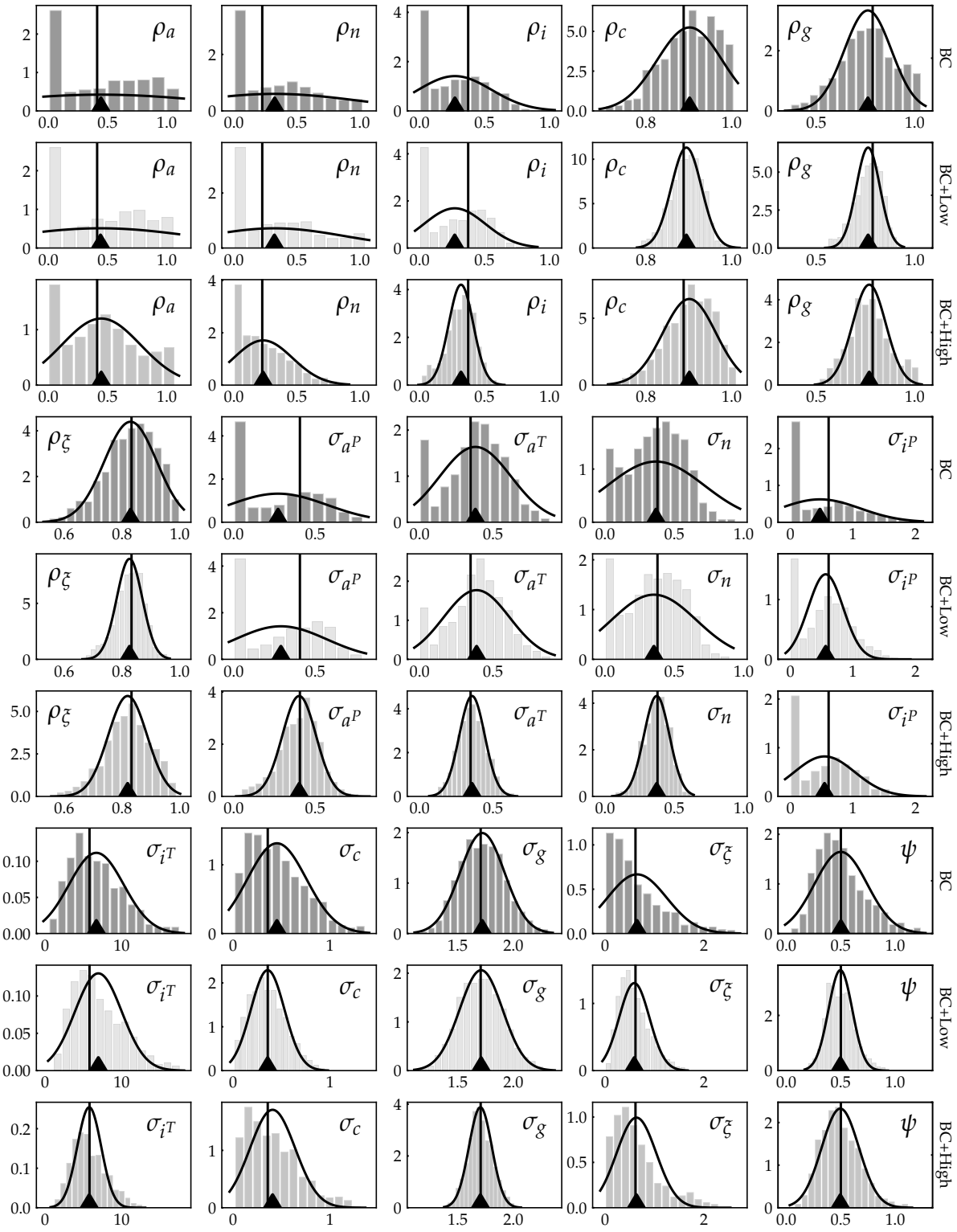


Figure 12: Joint estimation using the Whittle likelihood. The figure shows MC sampling distributions along with gaussian densities centered on the true values, with standard deviations equal to the marginal CRLB. The sample size is $T = 192$.

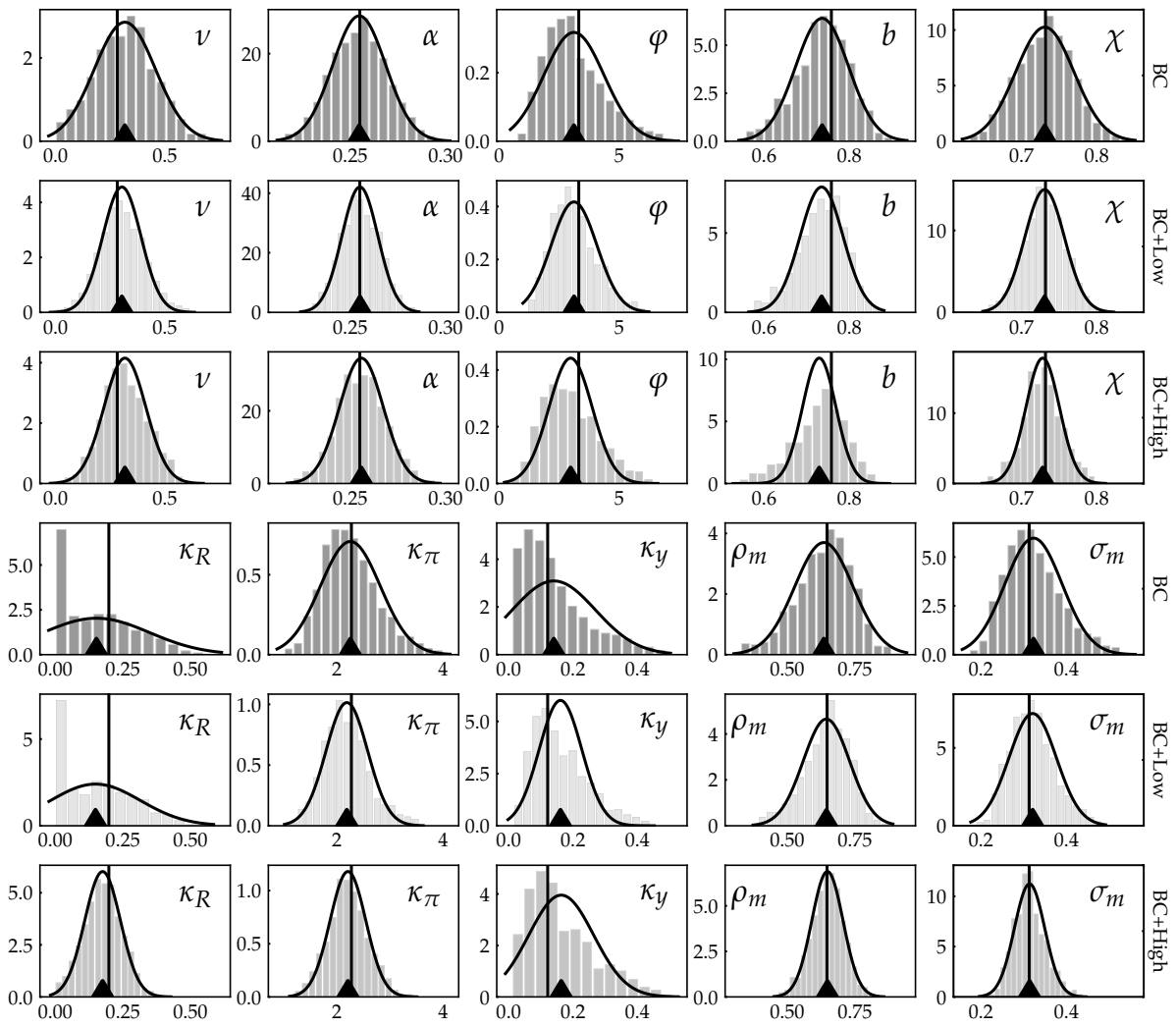


Figure 13: Continue Figure 12.

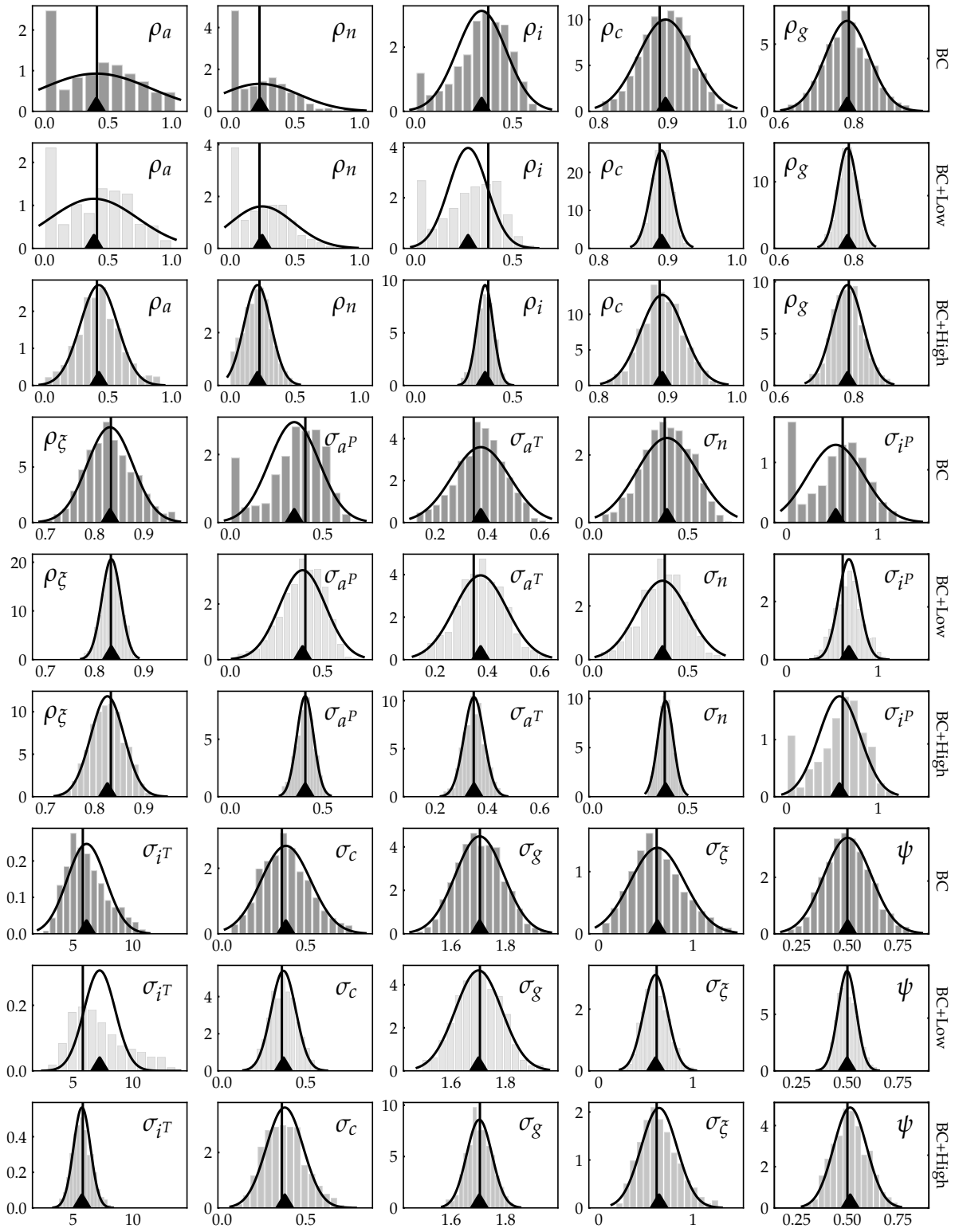


Figure 14: See the note to Figure 12. The sample size is $T = 1000$.

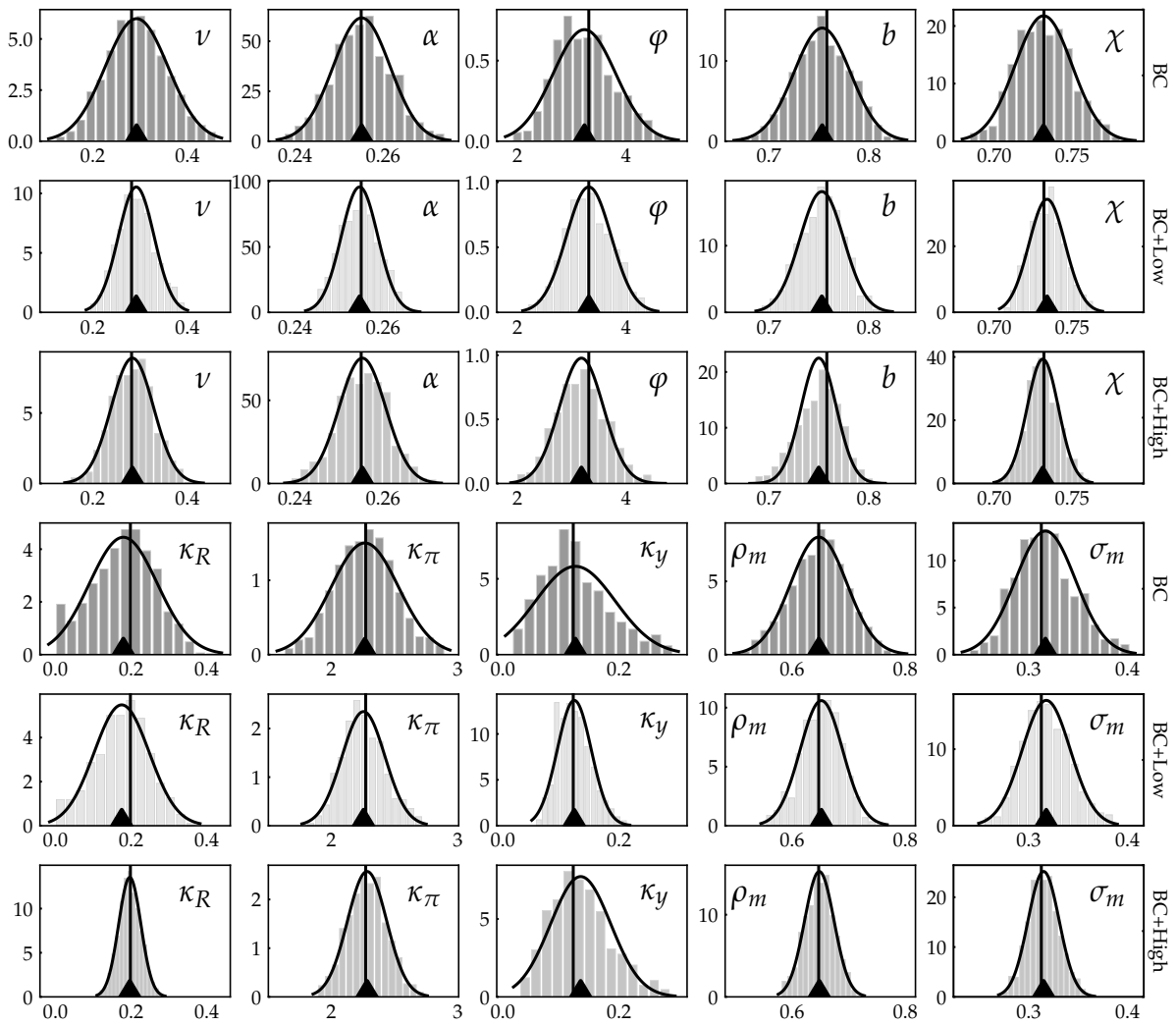
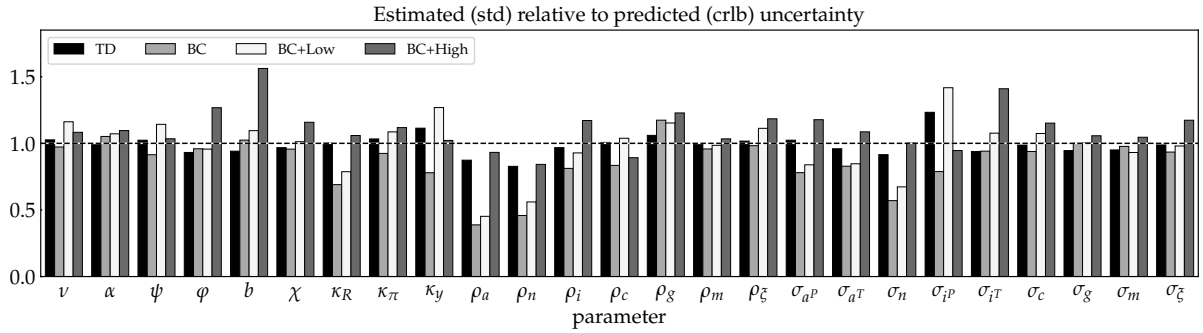
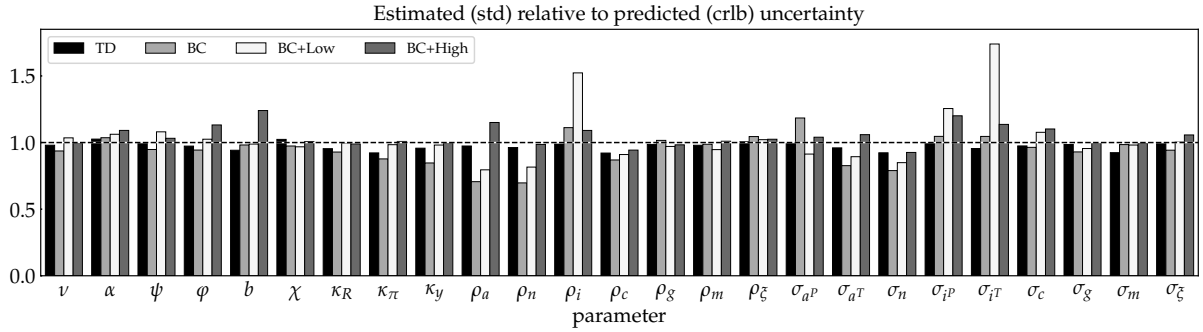


Figure 15: Continue Figure 14.



(a) T=192.

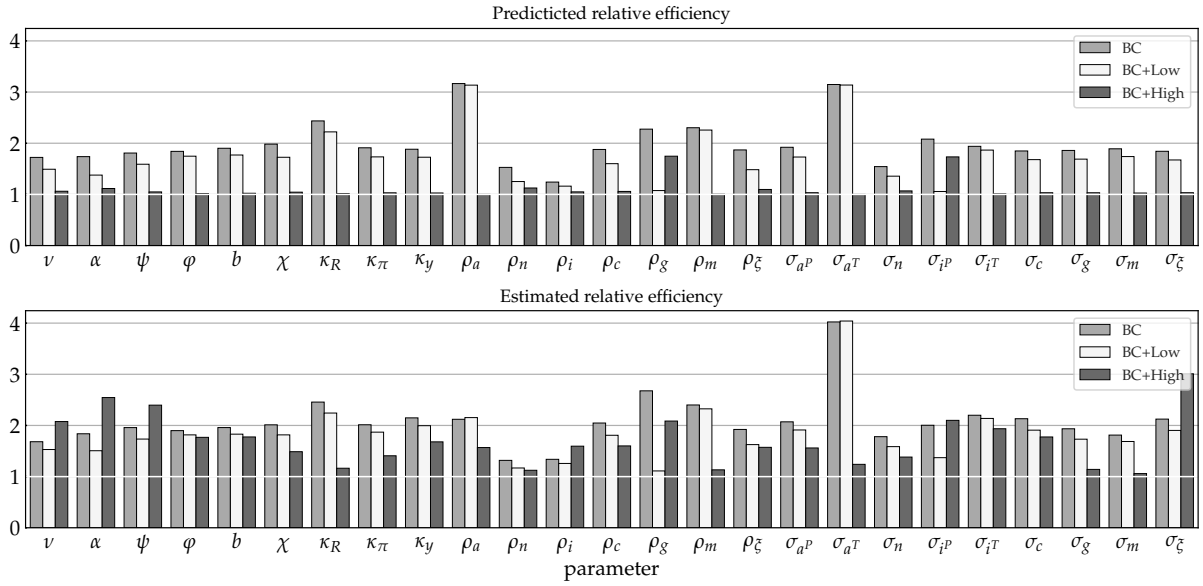


(b) T=1000.

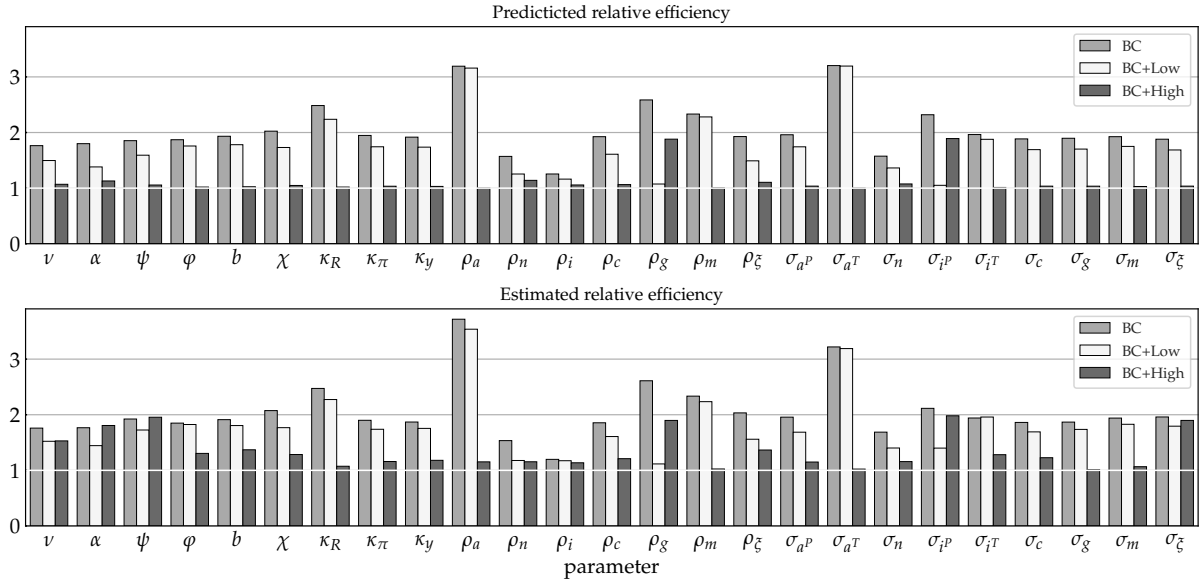
Figure 16: Ratios of Whittle-based unconditional MC standard deviations to unconditional CRLBs. TD represents the exact time domain MLE, while the other three are band spectral Whittle estimators using frequencies from the BC, BC+Low, and BC+High frequency bands.

D.2.2 Conditional estimation: BC vs BC+Low vs BC+High

The following results are for conditional estimation, where only one parameter is estimated at a time while the others are treated as known. The predicted efficiency loss is based on the conditional CRLBs in the band spectral and full information cases (see equation (6.1)).



(a) T=192.



(b) T=1000.

Figure 17: Predicted (top panel of each subplot) and MC-estimated (bottom panel) relative efficiency of the three band spectral estimators.

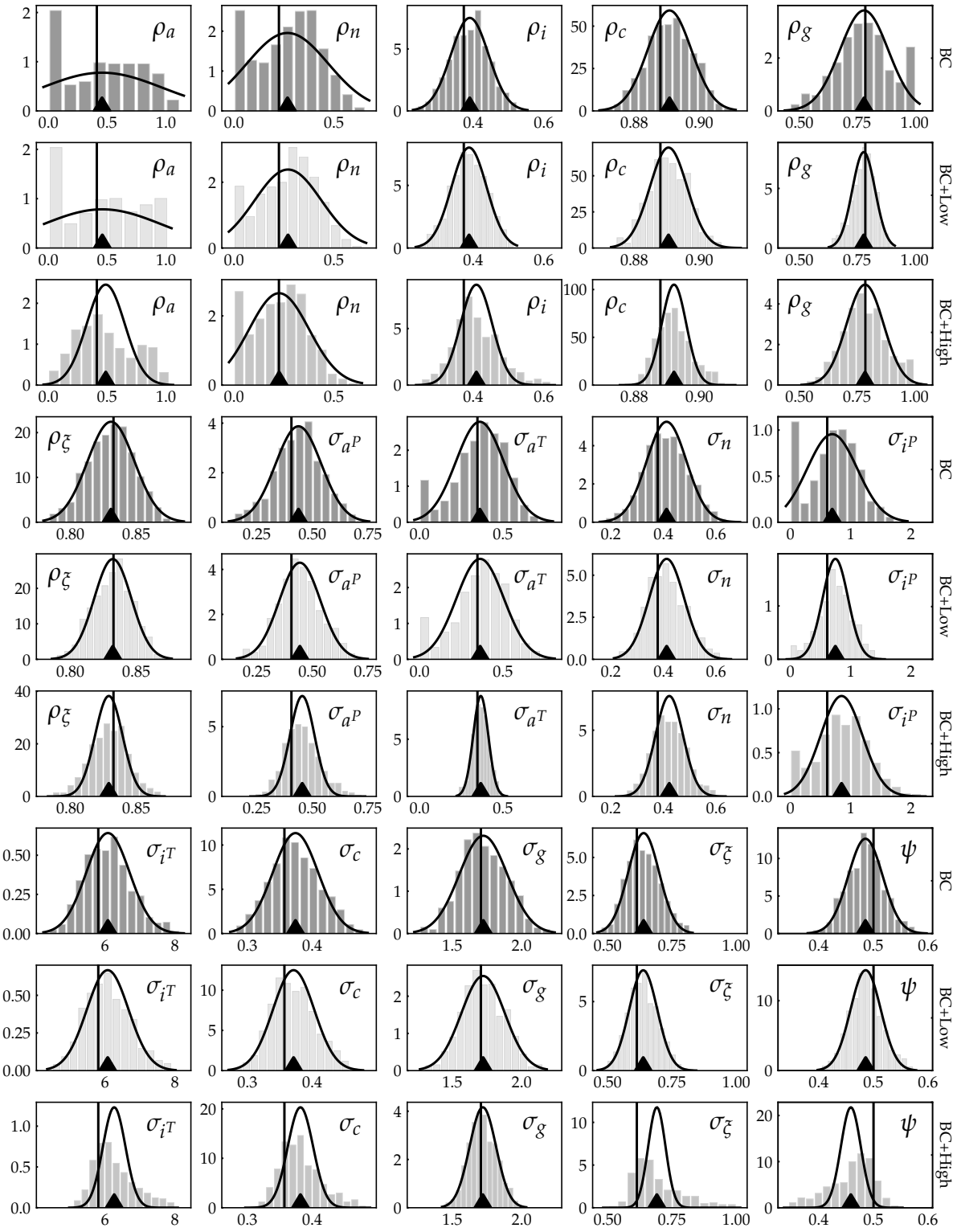


Figure 18: Conditional estimation using the Whittle likelihood. The figure shows MC sampling distributions and gaussian densities centered on the true values with standard deviation equal to the conditional CRLB. The sample size is $T = 192$.

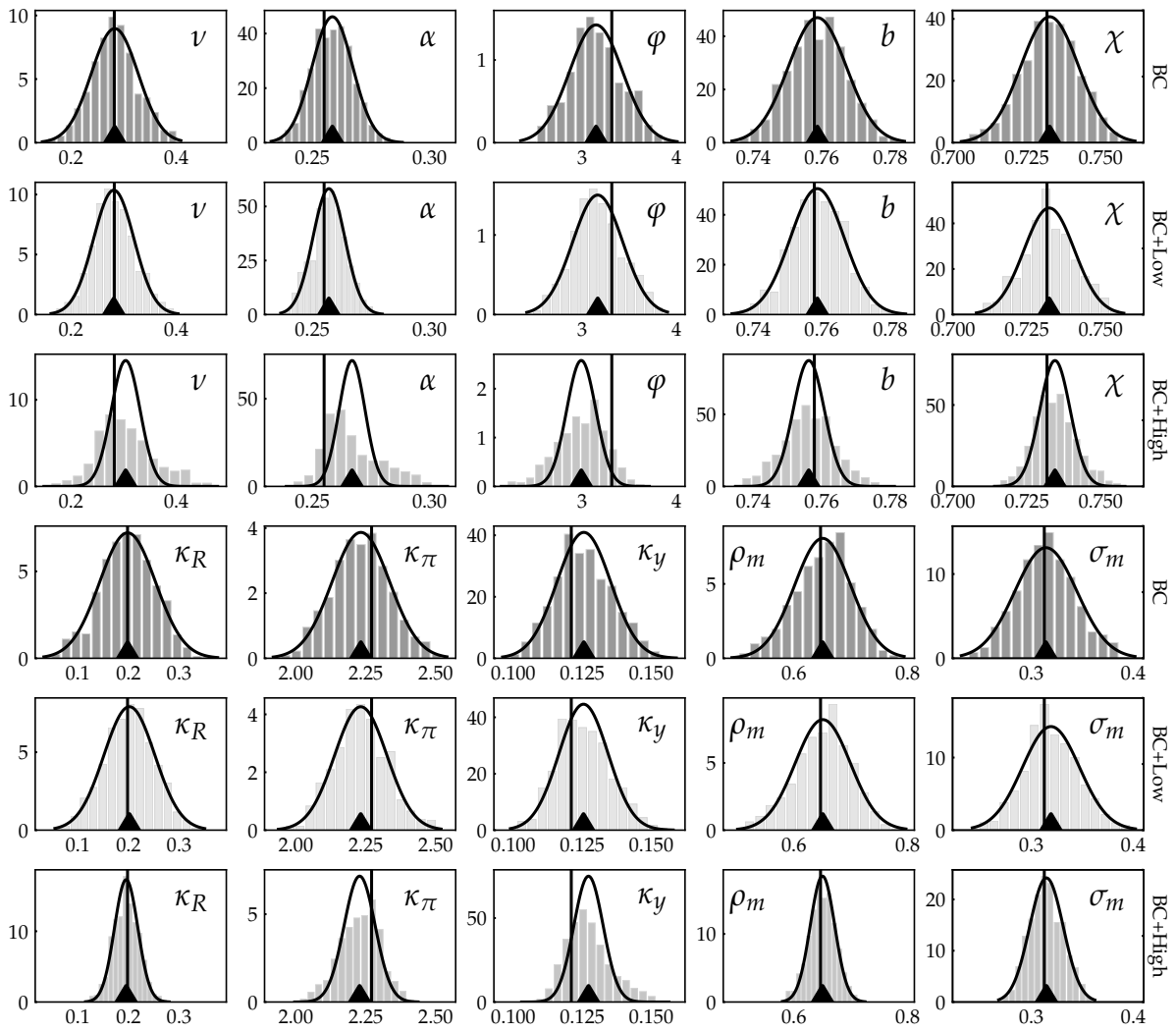


Figure 19: Continue Figure 18.

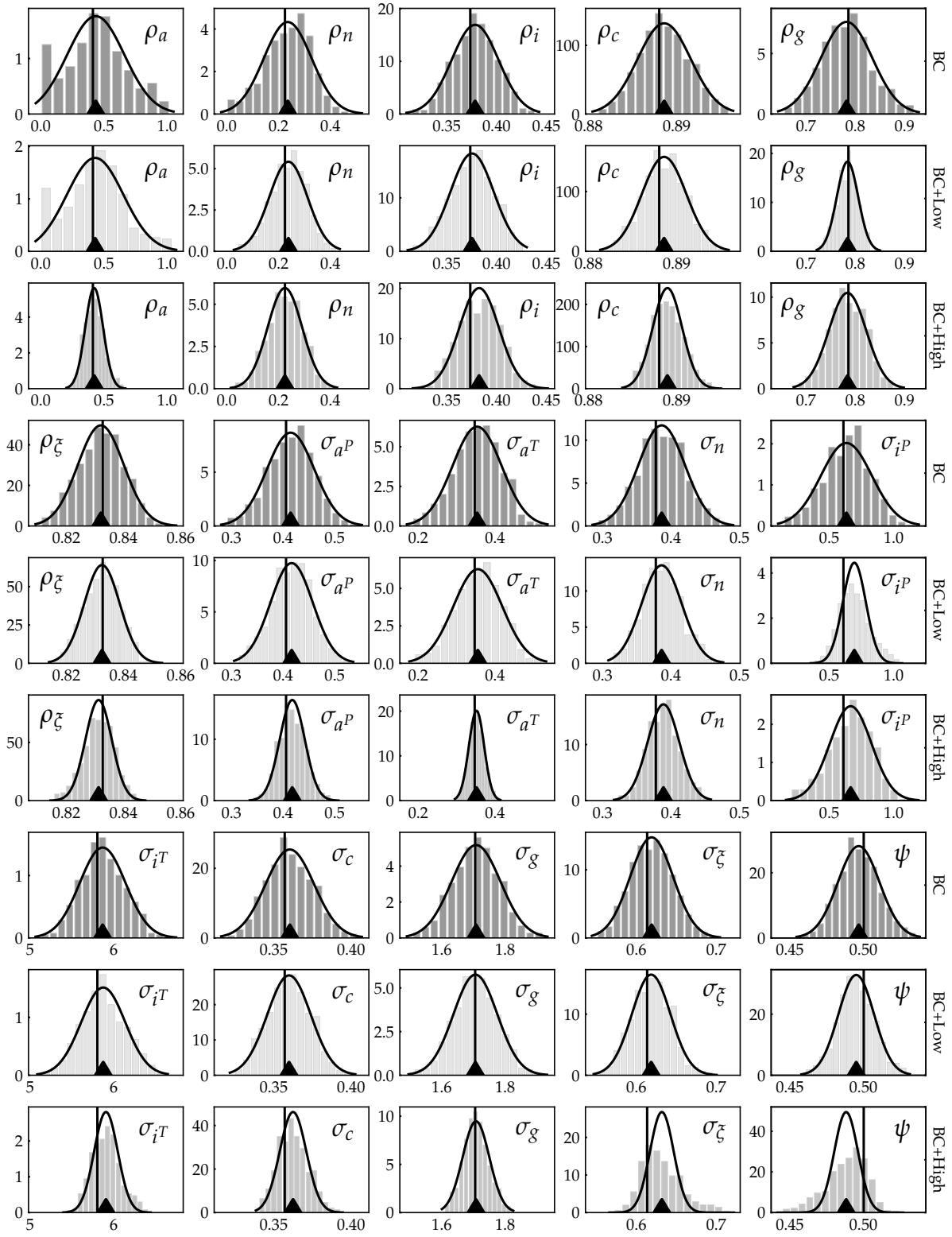


Figure 20: See the note to Figure 18. The sample size is $T = 1000$.

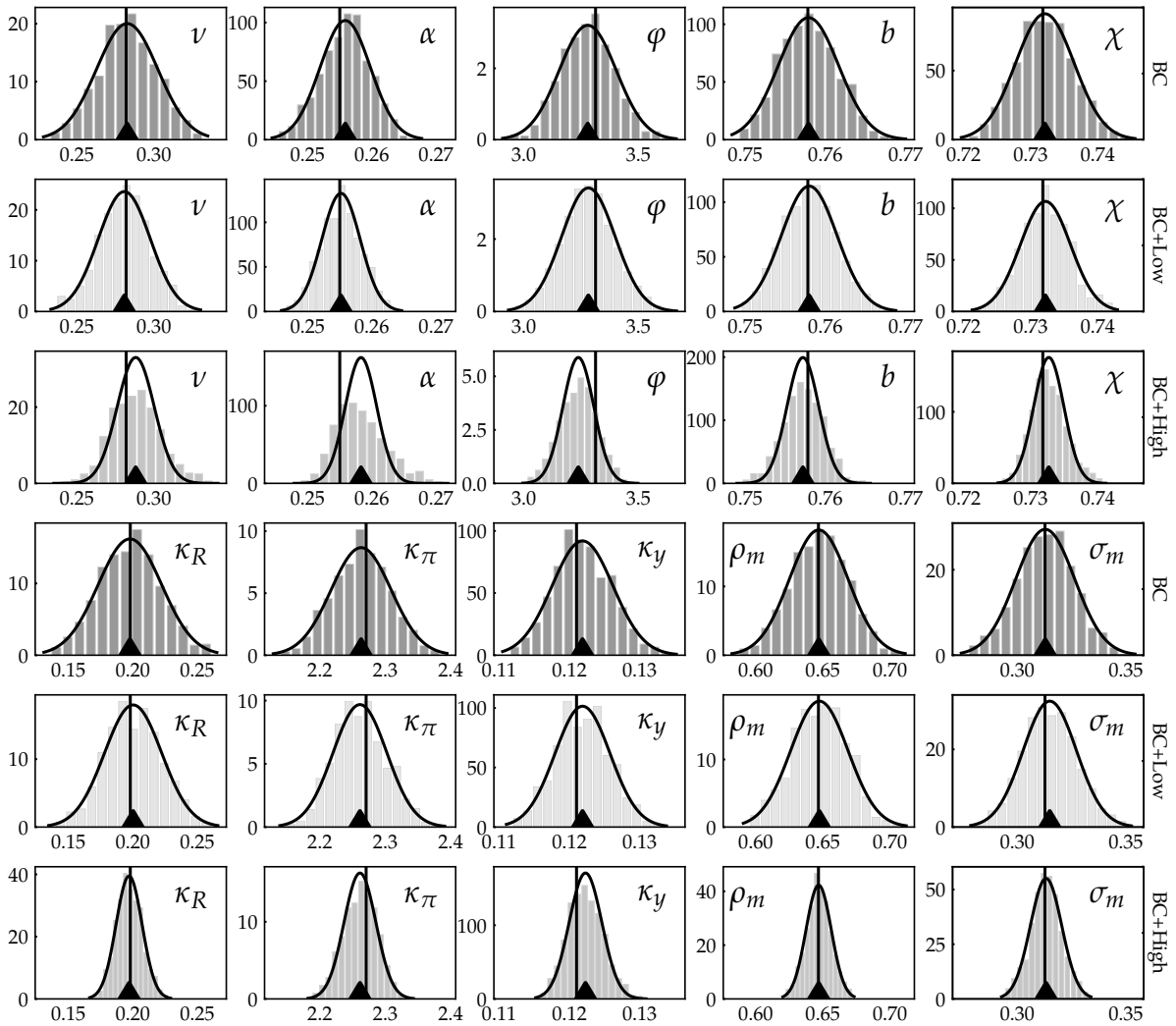
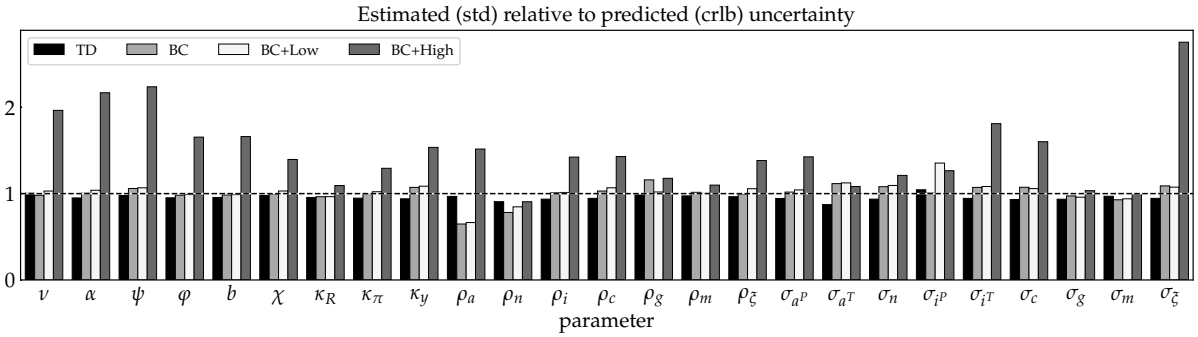
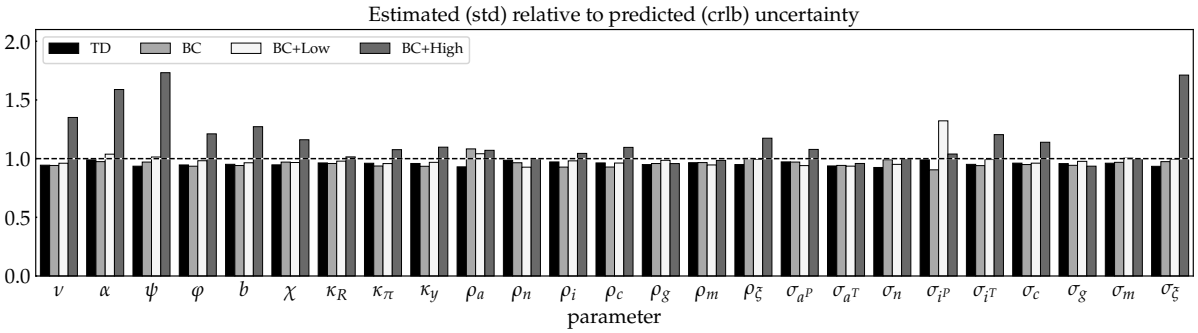


Figure 21: Continue Figure 20.



(a) T=192.

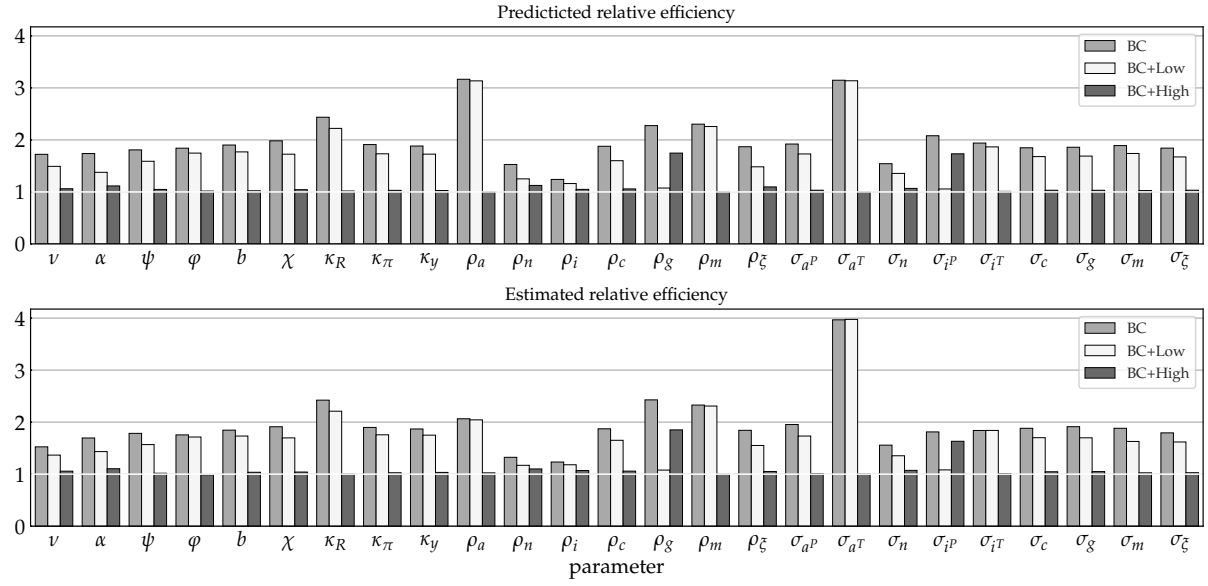


(b) T=1000.

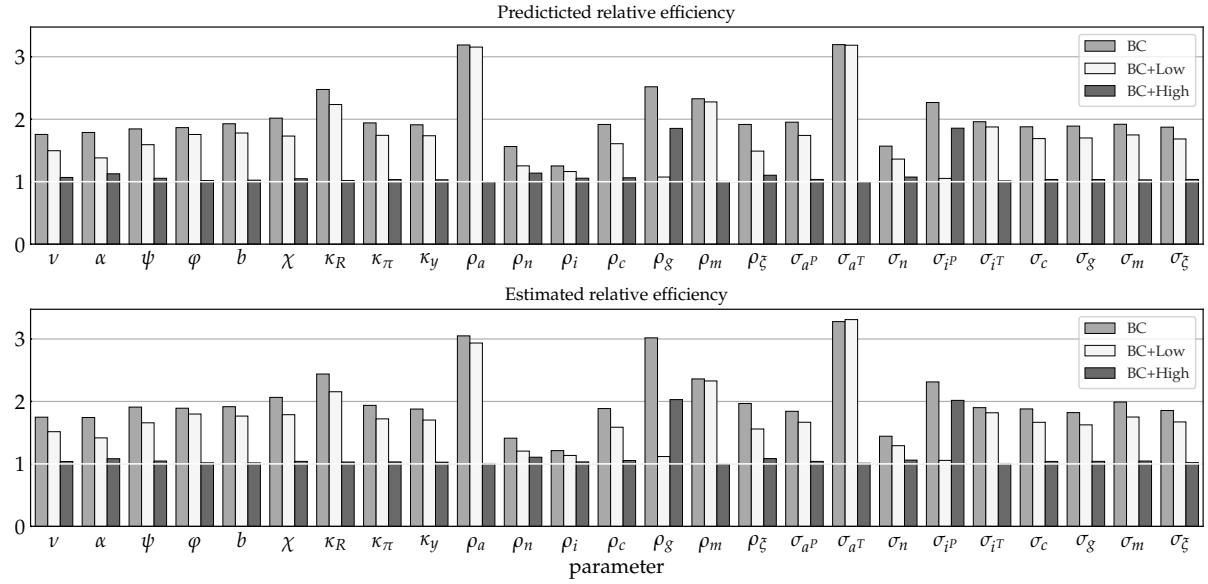
Figure 22: Ratios of Whittle-based conditional MC standard deviations to conditional CRLBs. TD represents the exact time domain MLE, while the other three are band spectral Whittle estimators using frequencies from the BC, BC+Low, and BC+High frequency bands.

D.2.3 Conditional estimation using the exact band spectral likelihood: BC vs BC+Low vs BC+High

The following results are for conditional estimation using the exact band spectral likelihood function instead of the Whittle approximation.



(a) T=192.



(b) T=500.

Figure 23: Predicted (top panel of each subplot) and MC-estimated (bottom panel) relative efficiency of the three band spectral estimators.

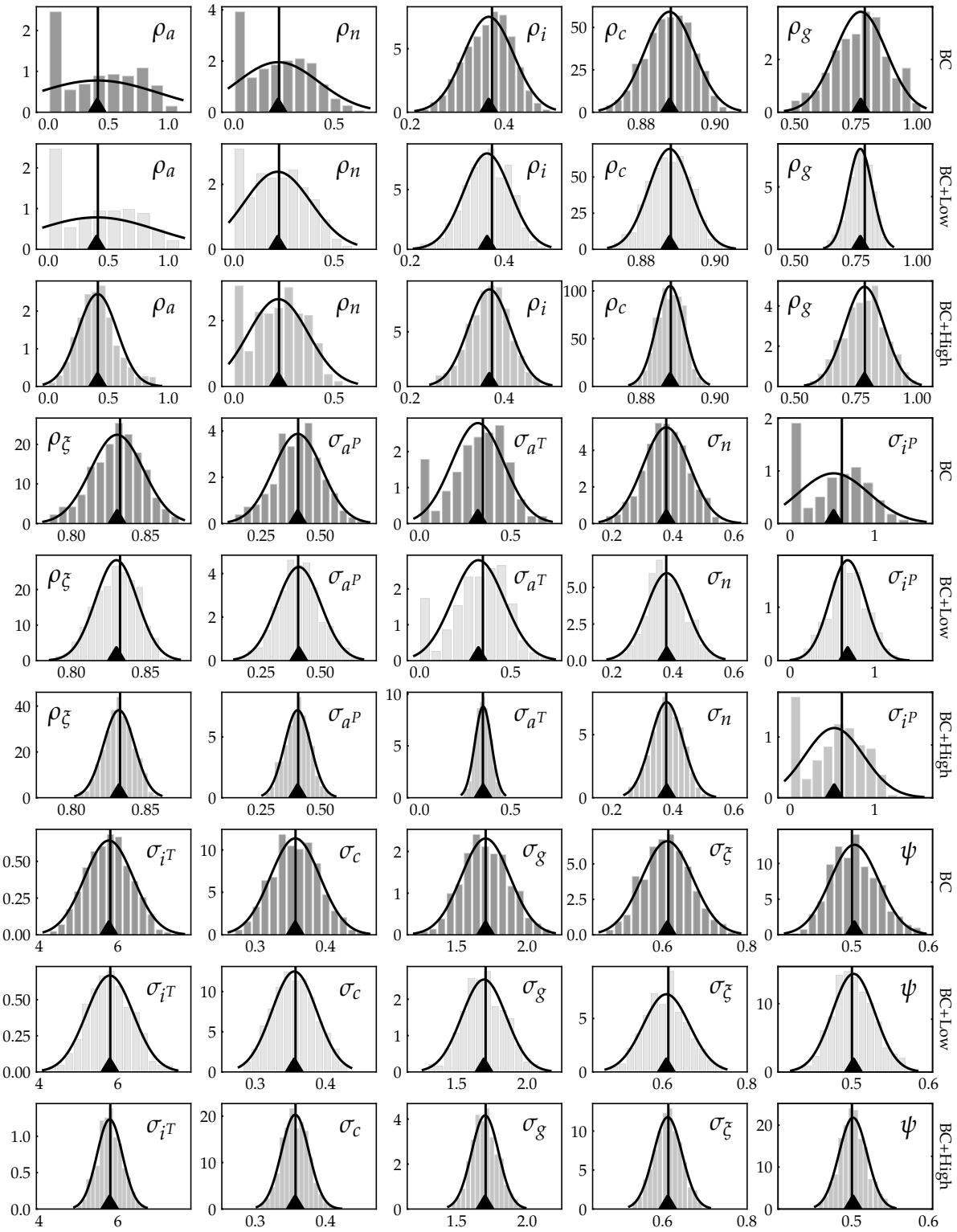


Figure 24: Conditional estimation using the exact likelihood. The figure shows MC sampling distributions and gaussian densities centered on the true values with standard deviation equal to the conditional CRLB. The sample size is $T = 192$.

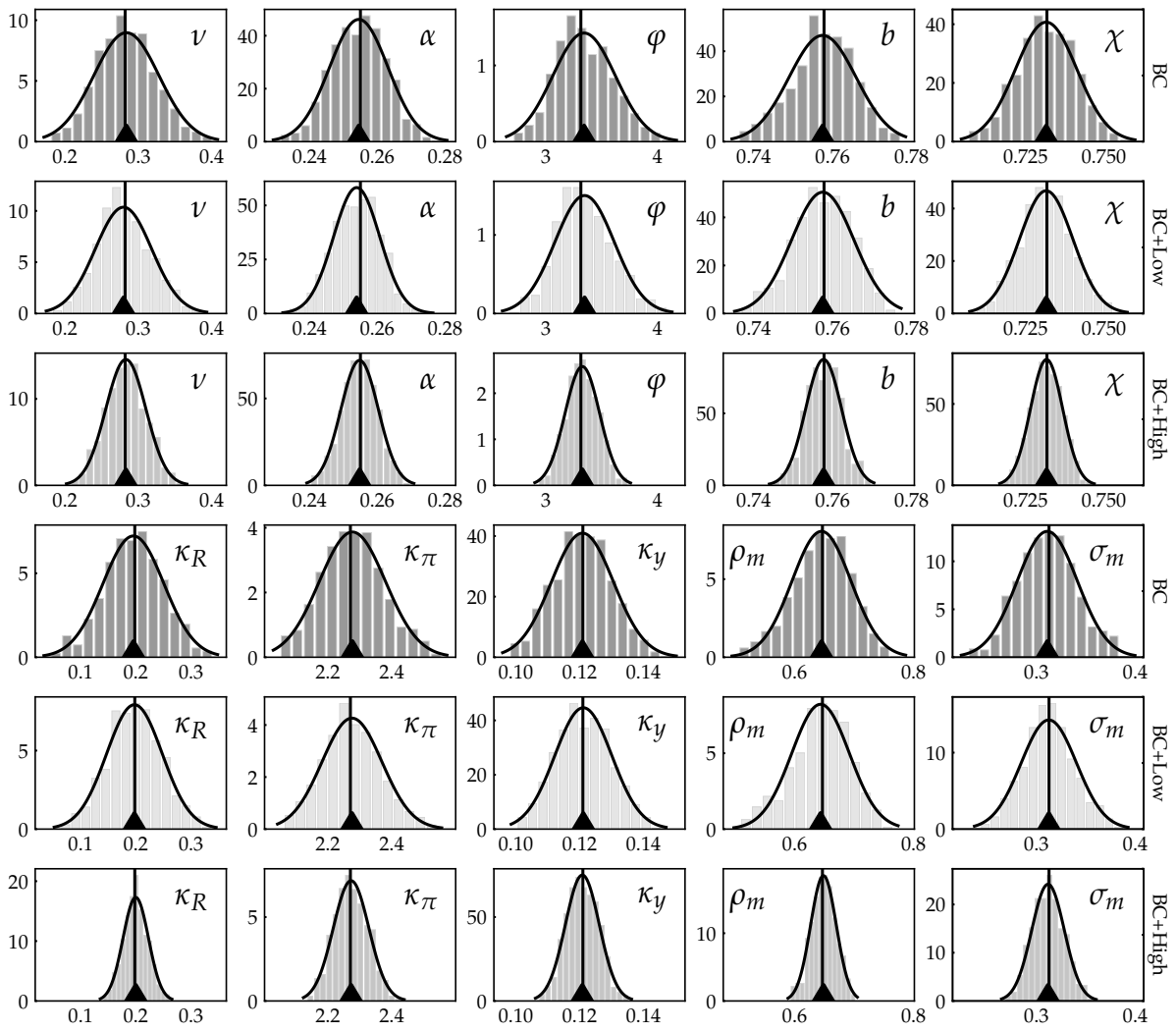


Figure 25: Continue Figure 24

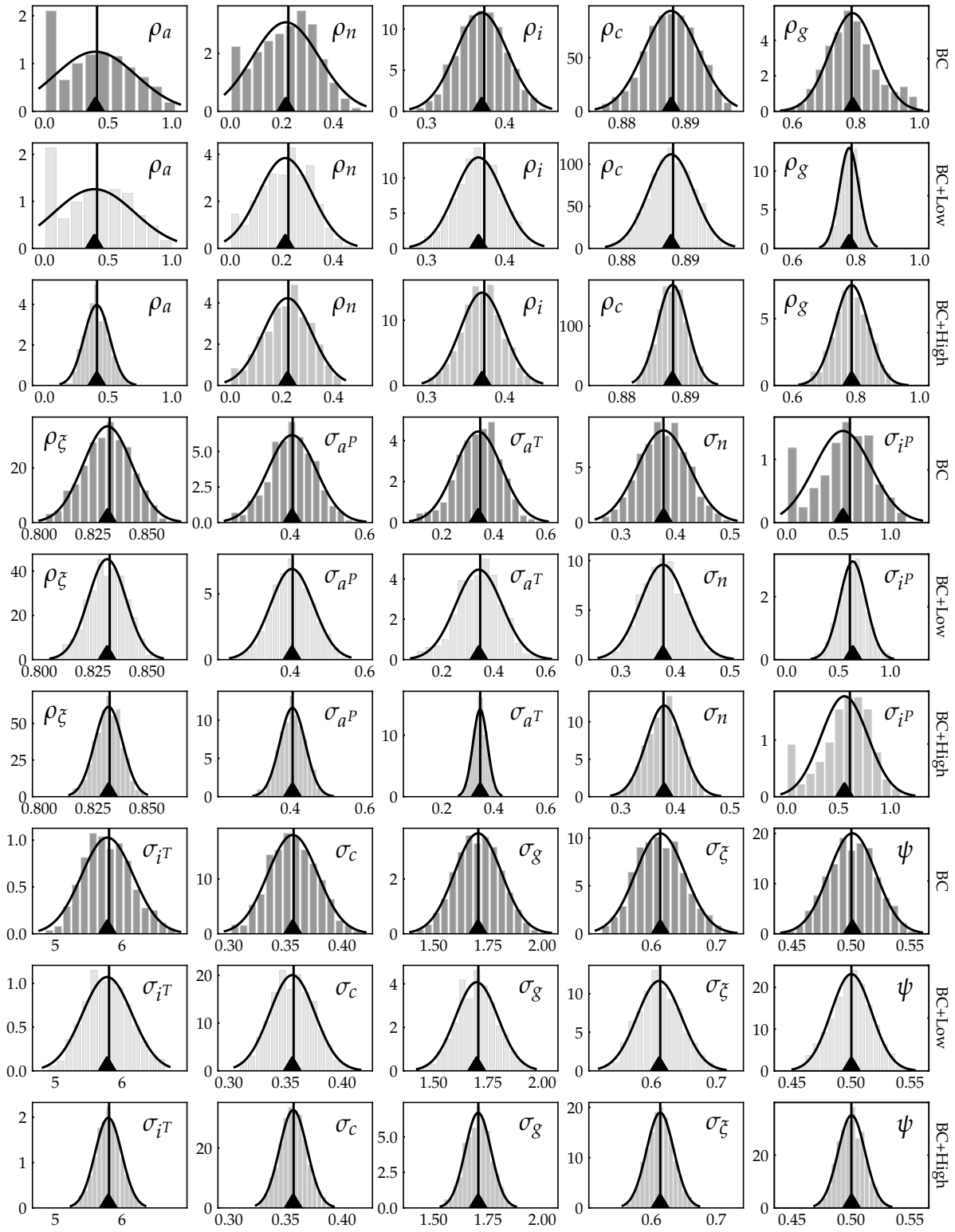


Figure 26: See the note to Figure 24. The sample size is $T = 500$.

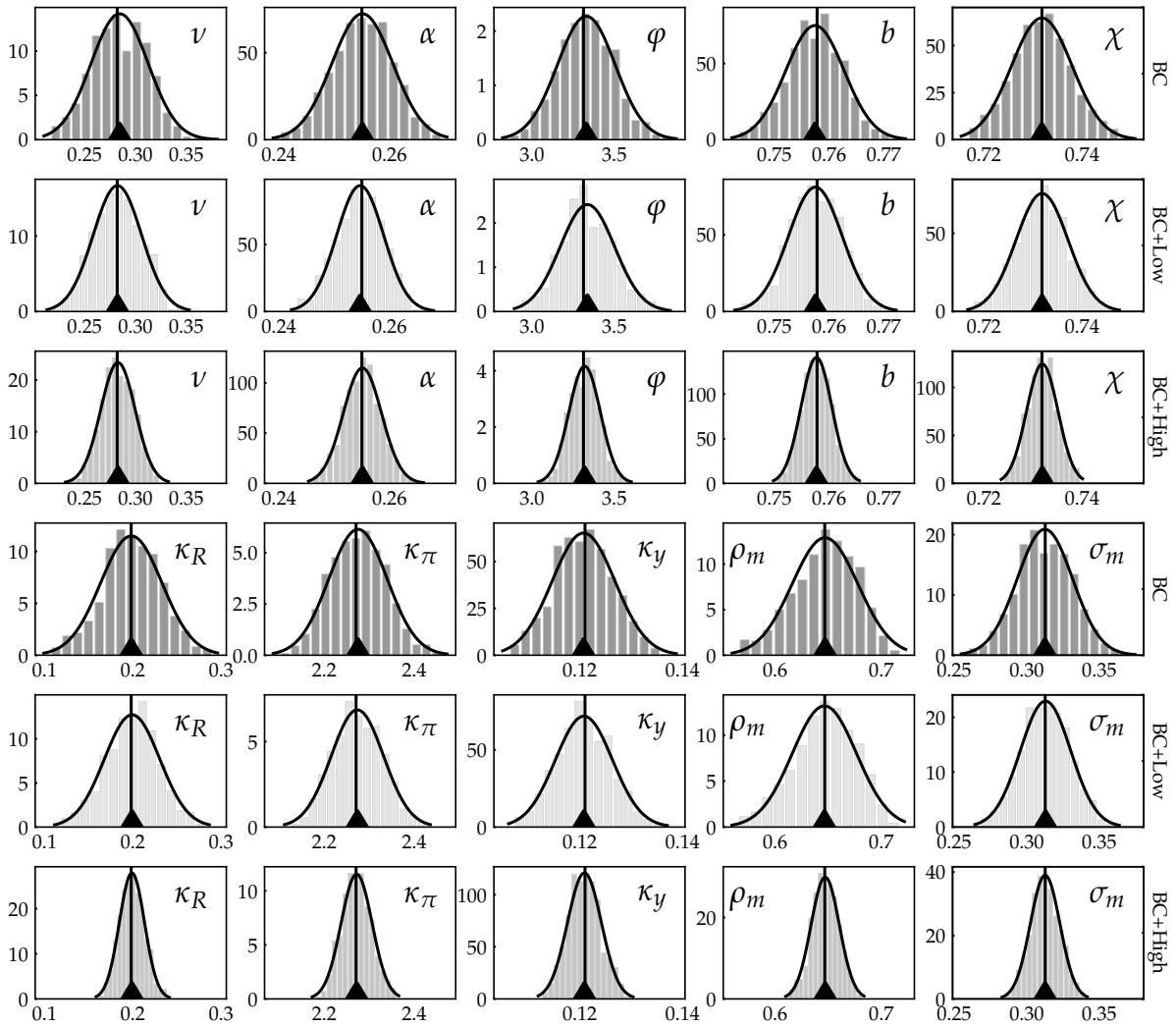
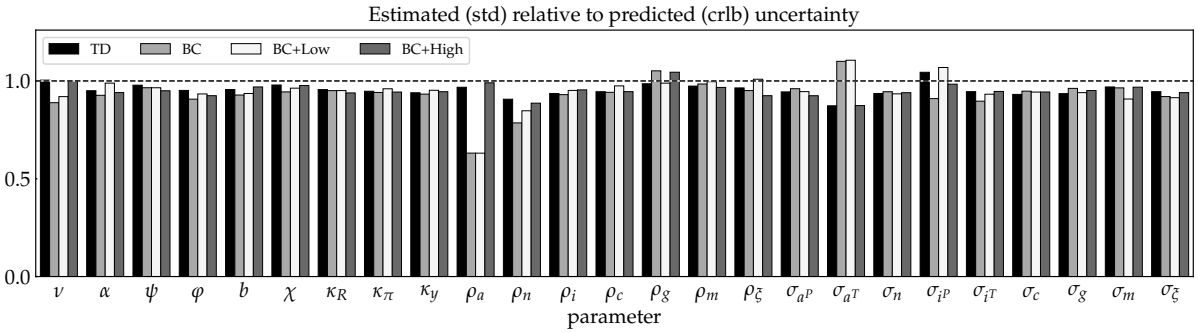
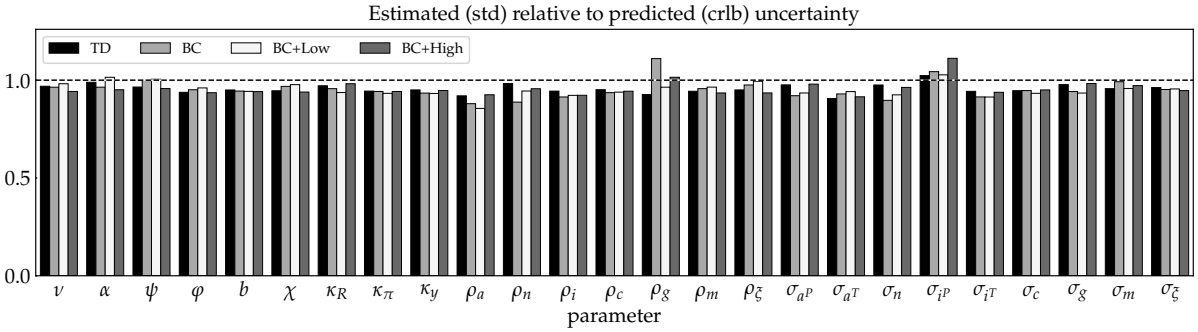


Figure 27: Continue Figure 26



(a) T=192.



(b) T=500.

Figure 28: Ratios of exact conditional MC standard deviations to conditional CRLBs. TD represents the exact time domain MLE, while the other three are exact band spectral likelihood estimators using frequencies from the BC, BC+Low, and BC+High frequency bands.

References

Angeletos, George-Marios, Fabrice Collard, and Harris Dellas (2018) "Quantifying confidence," *Econometrica*, 86 (5), 1689–1726.